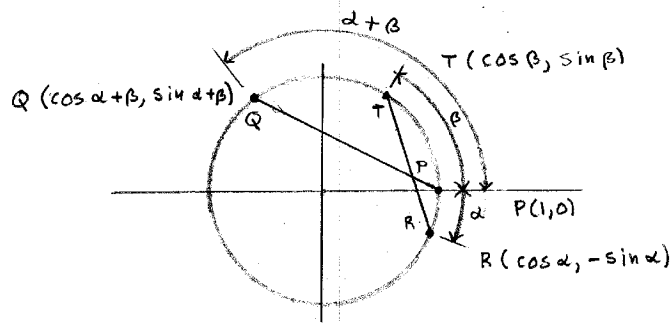


[09-11-14-RT-11]
Proof of addition theorem



Assume as in the figure above that the sum of an arc of length α and an arc of length β is an arc of length $\alpha + \beta$.

Since $\overset{\text{arc}}{PQ} = \overset{\text{arc}}{RT}$, the lengths chords PQ and RT are equal.

$$\begin{aligned} PQ^2 &= [\cos(\alpha + \beta) - 1]^2 + [\sin(\alpha + \beta) - 0]^2 \\ &= \cos^2(\alpha + \beta) - 2 \cos(\alpha + \beta) + 1 + \sin^2(\alpha + \beta) \\ &= 2 - 2 \cos(\alpha + \beta) \end{aligned}$$

$$\begin{aligned} RT^2 &= [\cos \beta - \cos \alpha]^2 + [\sin \beta + \sin \alpha]^2 \\ &= \cos^2 \beta - 2 \cos \alpha \cos \beta + \cos^2 \alpha + \sin^2 \beta + 2 \sin \alpha \sin \beta + \sin^2 \alpha \\ &= 2 - 2 \cos \alpha \cos \beta + 2 \sin \alpha \sin \beta \end{aligned}$$

Since $PQ = RT$

$$\begin{aligned} PQ &= RT \\ \Leftrightarrow 2 - 2 \cos(\alpha + \beta) &= 2 - 2 \cos \alpha \cos \beta + 2 \sin \alpha \sin \beta \\ \Leftrightarrow -2 \cos(\alpha + \beta) &= -2 \cos \alpha \cos \beta + 2 \sin \alpha \sin \beta \\ \Leftrightarrow \cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \end{aligned}$$

Therefore,

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

□